Wallace Hall Academy
Physics Department

Advanced Higher Physics

Rotational Motion

Solutions

TUTORIAL 1.0

Equations of motion

1. (a) $v = \frac{ds}{dt}$, $\frac{ds}{dt}$ = 90 – 8t, v = 90 – 8t
   (b) 0 = 90 – 8t t = 11 s
   (c) $a = \frac{dv}{dt}$, $\frac{dv}{dt}$ = -8, a = -8 m s^{-2}

2. $a = \frac{dv}{dt}$, $\int dv = \int a dt$, v = at + k
   when t = 0, k = u, where u is the velocity at t= 0
   giving v = u + at [you must show clearly that the constant of integration is u]

3. $\frac{ds}{dt}$ = u + at $\int ds = \int (u + at) dt$
   s = ut + ½ at^2 + k, at t = 0 s = 0, the origin of displacement is taken at t = 0.
   hence k = 0 and s = ut + ½ at^2 [you must show clearly that k = 0]

4. s = 8 – 10t + t^2
   $\frac{ds}{dt}$ = -10 + 2t and $\frac{d^2s}{dt^2}$ = 2, the acceleration is 2 units
   The acceleration is constant. It does not depend on time t.
   Hence the unbalanced force is constant, F = ma, and the mass m is constant.

5. (a) $\frac{ds}{dt}$ = $9t^2$ + 5, $\frac{d^2s}{dt^2}$ = 18t, a = 18t
   (b) The acceleration is a function of time, so it is not constant.

6. (a) $\frac{ds}{dt}$ = 5 + 2t, v = 5 + 2t
   (b) $\frac{d^2s}{dt^2}$ = 2, a = 2 m s^{-2}
   (c) 6 = 5t + $2t^2$ 0 = t^2 +5t –6 = (t + 6)(t -1) hence t = 1 s [t is positive]
   (d) v = 5 + 2t velocity = 7 m s^{-1} at t = 1

7. $\int dv = \int a dt$ giving v = at + k, at t = 0 v = 3 given, hence k = 3,
   thus v = 4t + 3, a = 4 given,

8. $\int ds = \int (2 + 6t) dt$
   s = 2t + $3t^2$ +k. assume s = 0 at t = 0 hence k = 0
   s = 2t +$3t^2$

9.

(a)

Horizontally	Vertically
	Let upwards be positive direction
$u_{horiz} = 20 \cos 30° = 17.3 \text{ m s}^{-1}$	$u_{vert} = 20 \sin 30° = 10 \text{ m s}^{-1}$
	$s_{vert} = +30 \text{ m}$
	$a = -9.8 \text{ m s}^{-2}$

Consider the vertical motion to find the time

$$s = ut + \frac{1}{2} a t^2 \quad \text{directly.}$$

$$30 = 10\,t - \frac{1}{2}\, 9.8\, t^2$$

$$4.9\, t^2 - 10\, t - 30 = 0$$

$$t = \frac{+10 \pm \sqrt{100 + 4 \times 4.9 \times 30}}{2 \times 4.9} \quad \text{(using the quadratic formula)}$$

$t = \underline{\underline{3.7}}$ s (the negative solution is not applicable)

Alternative solution

Consider the vertical motion in two parts. Let upwards be positive direction.
Time from start to highest point using $v = u + at$ gives $t = 1.02$ s
Time from highest point to the ground:

height above 30 m using; $s = \frac{(u + v)}{2} t = 5.1$ m, total height = 35.1 m.

Hence time from highest point using $s = ut + \frac{1}{2} a t^2$ is $t = 2.68$ s

Thus total time to reach the ground = 1.02 + 2.68 = $\underline{\underline{3.7}}$ s (to 2 sig. figs.)

TUTORIAL 2.0

Angular motion

1. 360° = 2π radians, hence to convert from degrees to radians multiply by $\frac{\pi}{180}$

$\pi/6$ (0.52), $\pi/4$(0.78), $\pi/2$(1.57), π(3.14), 1.5 x π(4.71), 2π(6.28), 4π(12.6)
[Notice that 360° is 6.28 radians, it is useful to get a feel for the size of a radian]

2. To convert from radians to degrees multiply by $\frac{180}{\pi}$.

57.3°, 573°, 5.73°, 180°, 360°, 90°, 30° [Notice 1 rad is just under 60°]

3. One revolution is 2π radians. Hence 1 revolution per minute is $\frac{2\pi}{60}$ rad s^{-1}.

3.46 rad s^{-1}, 4.71 rad s^{-1}, 8.17 rad s^{-1}, 31.4 rad s^{-1}

4. (a) $\omega = \frac{d\theta}{dt}$ (b) $\alpha = \frac{d\omega}{dt}$ (c) $\alpha = \frac{d^2\theta}{dt^2}$

5. $\omega = \omega_0 + \alpha t$, $\theta = \omega_0 t + ½\alpha t^2$, $\omega^2 = \omega_0^2 + 2\alpha\theta$
where □ is the angular velocity at time t, ω_0 is the initial angular velocity at t = 0 and θ the angular displacement.

6. (a) $\omega = \omega_0 + \alpha t$, ω = 200 – 5 x 4 gives ω = 180 rad s^{-1}
(b) $\theta = \omega_0 t + ½\alpha t^2$, θ = 200 x 4 - ½ x 5 x 4^2 gives θ = 760 rad

7. (a) $\omega = \omega_0 + \alpha t$, $\frac{3000 \text{ x } 2\pi}{60} = \frac{800 \text{ x } 2\pi}{60} + 8\alpha$ gives α = 29 rad s^{-2}

(b) $\theta = \omega_0 t + ½\alpha t^2$, $\theta = \frac{800 \text{ x } 2\pi \text{ x } 8}{60}$ + ½ x 29 x 8^2 gives θ = 1.60 x 10^3 rad

(c) 255 revolutions

8. (a) (i) $\omega = \omega_0 + \alpha t$, ω = 0 + 3 x 5, ω = 15 rad s^{-1}
(ii) $\theta = \omega_0 t + ½\alpha t^2$ θ = 0 + ½ x 3 x 25 = 37.5 rad
(b) v = rω = 1.5 x 15 = 22.5 m s^{-1}

9. (a) (i) Time for one revolution = 365 x 24 x 60 x 60 = 3.15 x 10^7 s
ω = 2.0 x 10^{-7} rad s^{-1}
(ii) Time for one revolution = 28 x 24 x 60 x 60 s ω = 2.6 x 10^{-6} rad s^{-1}
(iii) Time for one revolution = 24 x 60 x 60 s ω = 7.3 x 10^{-5} rad s^{-1}
(iv) ω = 7.3 x 10^{-5} rad s^{-1} since one revolution per day
(b) using v = rω for each of the above with the orbit radius in metres
(i) 2.0 x 10^{-7} x 1.5 x 10^8 x 10^3 = 30 000 m s^{-1}
(ii) 2.6 x 10^{-6} x 3.8 x 10^5 x 10^3 = 990 m s^{-1}
(iii) 7.3 x 10^{-5} x 6.4 x 10^3 x 10^3 = 470 m s^{-1}
(iv) 7.3 x 10^{-5} x 3.6 x 10^4 x 10^3 = 2600 m s^{-1}

10. Consider a particle moving with uniform speed in a circular path.

$$\omega = \frac{d\theta}{dt}$$

The rotational speed v is constant, ω is also constant.
T is the period of the motion and is the time taken to cover 2π radians.

$$\omega = \frac{2\pi}{T} \text{ but } v = \frac{2\pi r}{T}$$

$$\boxed{v = r\omega}$$

TUTORIAL 2.1

Angular motion

1 $$2\pi \text{ radians} = 360^\circ$$
$$1 \text{ radian} = \frac{360}{2\pi} = \underline{\underline{57.3^\circ}}$$

2 $\theta = 2\pi$ $\omega = \frac{\theta}{t} = \frac{2\pi}{60}$

$t = 60$ s $\omega = \underline{\underline{0.10}}$ rad s^{-1}

3 (a) angular displacement θ = "area" under ω,t graph

$$\text{total area} = (5 \times 3) + \frac{1}{2}(3 \times 5) = 15 + 7.5$$
$$\theta = \underline{\underline{22.5}} \text{ radians}$$

(b) Similar calculation to (a) $\theta = \underline{\underline{60}}$ radians

(c) $$\alpha = \frac{\omega - \omega_o}{t} = \frac{15 - 5}{6}$$
$$\alpha = \underline{\underline{1.67}} \text{ rad s}^{-2}$$

4 (a) 100 revs = 100 x 2π radians; 1 min = 60 s

$$100 \text{ r.p.m.} = \frac{100 \times 2\pi}{60} = \underline{\underline{10.5}} \text{ rad s}^{-1}$$

(b) $$\alpha = \frac{\omega - \omega_o}{t} = \frac{10.5 - 0}{12} = \underline{\underline{0.88}} \text{ rad s}^{-1}$$

5 (a) $$\alpha = \frac{\omega - \omega_o}{t} = \frac{300 - 100}{10} = \underline{\underline{20}} \text{ rad s}^{-2}$$

(b) $$\theta = \omega_o t + \frac{1}{2}\alpha t^2$$
$$= (100 \times 10) + \frac{1}{2} \times 20 \times (10)^2$$
$$= \underline{\underline{2000}} \text{ radians}$$

(c) distance = circumference x no. of revs

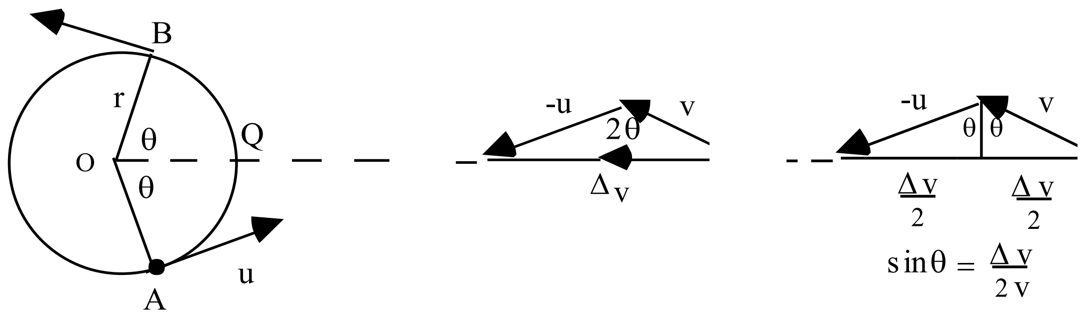
$$= 2\pi r \times \frac{\theta}{2\pi} = r\theta$$
$$\text{distance} = 0.12 \times 2000$$
$$= \underline{\underline{240}} \text{ m}$$

TUTORIAL 3.0

Central force

1. (a) $a_r = r\omega^2$
   (b) angular acceleration has the unit rad s^{-2}, radial acceleration has the unit m s^{-2}
   (c) angular acceleration is the rate of change of angular velocity radial acceleration is the rate of change of linear velocity which is directed to the centre of the circular motion

2.



The particle travels from A to B in time Δt and with speed v, thus | u | = | v | and Δv = v + (−u) which is Δv = v − u

$$\Delta t = \frac{arc\ AB}{v} = \frac{r(2\theta)}{v}$$

average acceleration, $a_{av} = \frac{\Delta v}{\Delta t} = \frac{2v\,sin\theta}{\Delta t}$

$$= \frac{2v\,sin\theta}{r\,2\theta/v} = \frac{v^2 sin\theta}{r\theta}$$

As θ tends to 0, a_{av} tends to the instantaneous acceleration at point Q:

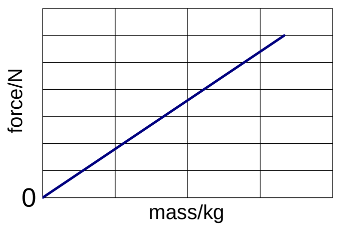
$a = \frac{v^2}{r} \cdot [\lim_{\theta\to 0} \frac{\sin\theta}{\theta}]$ but $[\lim_{\theta\to 0} \frac{\sin\theta}{\theta}] = 1$

when θ is small and is measured in radians sin $\theta = \theta$.

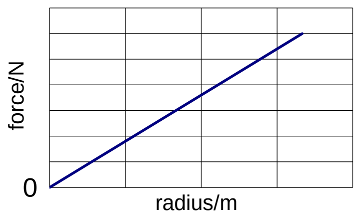
$$a = \frac{v^2}{r} = \omega^2 r \quad \text{since} \quad v = r\omega$$

3.

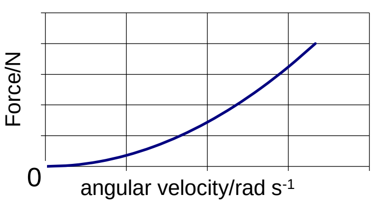
Force against mass



Force against radius



Force against angular velocity

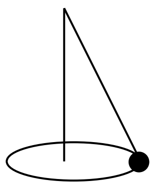


4. (a) $F = mr\omega^2$ $56 = 0.15 \times 1.2 \times \omega^2$ giving $\omega = 18$ rad s^{-1}

(b) 85 rpm = (85 x 2 π/ 60) rad s-1

$F = mr\omega^2$ $56 = 0.15 \times r \times \left(\frac{85 \times 2 \times \pi}{60}\right)^2$ giving r = 4.7 m

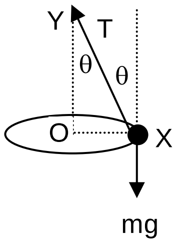
5. (a)



(b) $a = \frac{v^2}{r} = \frac{1.33^2}{0.5}$

$a = 3.5$ m s^{-2}

(c)



Central force is T sinθ

$T \sin\theta = m\frac{v^2}{r}$ (since $\sin\theta = \frac{OX}{XY}$)

$T \times \frac{0.5}{1.5} = 0.2 \times 3.5$

T = 2.1

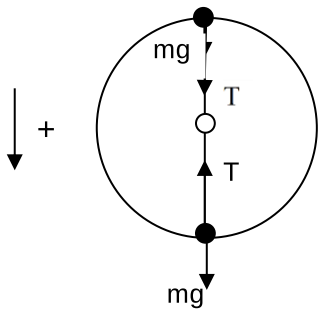
6. (a) At the top

$T + mg = m\frac{v^2}{r}$

$T = m\frac{v^2}{r} - mg$

$= 3.0 \times \frac{8.0^2}{0.75} - 3 \times 9.8$

Tension at top = 227 N



(b) At bottom

$mg + (-T) = m\frac{-v^2}{r}$

$T - mg = m\frac{v^2}{r}$

Tension at bottom = 285 N

7. Weight of car = mg and central force $= m\frac{v^2}{r}$

Car will leave road if central force just equals weight of car

hence greatest speed is when $g = \frac{v^2}{r}$ and v = 18.5 m s^{-1}

8. (a) $F = m r\omega^2 = m \times 5 \times (20 \times \frac{2\pi}{60})^2$ hence F/m = 22 N kg^{-1} = 2.2g

(b) F/m = 3 x 9.8 = 5 x (Rot rate x $\frac{2\pi}{60}$)^2 thus rotational rate = 23 rpm.

9. The centripetal force on an object undergoing circular motion is the inwards force towards the centre of the circle which keeps the object rotating in a circle.
The centrifugal force is the apparent force in an outward direction on a rotating object.

TUTORIAL 3.1

Circular Motion

1 (a) At this distance the central force is provided by gravitational attraction.

$g = 7.0 \text{ m s}^{-2}$ $\qquad \frac{m v^2}{r} = m g$; $\qquad$ m cancels

$$v^2 = g r = 7.0 \times 7.5 \times 10^6 = 7.246$$

Thus $v = \underline{7.2 \times 10^3} \text{ m s}^{-1}$

(b) $v = \frac{2 \pi r}{T} = \frac{2 \pi \times 7.5 \times 10^6}{T}$

$$T = \frac{2 \pi \times 7.5 \times 10^6}{7.2 \times 10^3} = \underline{6.5 \times 10^3} \text{ s}$$ (or $T = 7.25 \times 10^3$ may have been used)

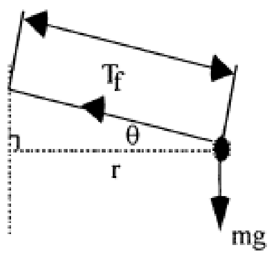
2 This means that the Earth would be rotating so quickly that **all** of the body's weight is required to supply the central force.

$mg = \frac{m v^2}{r}$ and $\frac{(2 \pi r/T)^2}{r} = \frac{4 \pi^2}{T^2}$ $\qquad$ m cancels

$$T^2 = \frac{4 \pi^2 r}{g} = \frac{4 \pi^2 \times 6.4 \times 10^6}{9.8}$$

$$T = 5.08 \times 10^3 \text{ s} = \underline{85} \text{ minutes}$$

3



T_f = tension

T = period

(a) Here we are allowed to assume $\theta = 0°$ and $r = L$

T_f supplies central force $= \frac{mv^2}{r}$

$$T_f = \frac{m (2\pi r/T)^2}{r} = \frac{m 4 \pi^2 r}{T^2} = \frac{0.20 \times 4 \pi^2 \times 0.80}{0.25^2}$$

Thus tension = $\underline{101}$ N

(b) In practice $\theta \neq 0$ because the weight will always pull the string down unless the orbital speed is infinite.

(c) Resolving forces: $\quad T_f \cos\theta = \frac{m v^2}{r} = \frac{m 4 \pi^2 r}{T^2}$ and

$T_f \sin\theta = mg$ (and $r = L \cos\theta$)

Dividing: $\quad \frac{T_f \sin\theta}{T_f \cos\theta} = \frac{mg \times T^2}{m 4 \pi^2 L \cos\theta}$

$\sin\theta = \frac{g \times T^2}{4 \pi^2 L}$ and $\sin\theta = \frac{9.8 \times 0.25^2}{4 \pi^2 \times 0.80}$

thus $\theta = \underline{1.1}°$

4 The gravitational attraction supplies the central force.

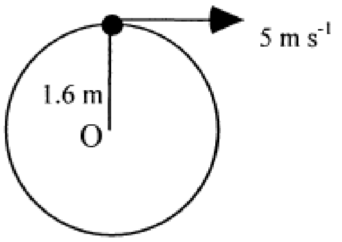
$$m\,g = \frac{m\,v^2}{r} \qquad \text{m cancels}$$

$$g = \frac{(2\pi r/T)^2}{r} = \frac{4\pi^2 r}{T^2} = \frac{4 \times \pi^2 \times 4.0 \times 10^8}{(2.0 \times 10^6)^2} \quad (\text{Using } T = 2.0 \times 10^6 \text{ s})$$

Thus the value of "g" at the moon's orbit = $\underline{\underline{3.9 \times 10^{-3}}}$ m s^{-2}

(You may have noticed that 27.3 days is 2.36×10^6 s. The 2.0×10^6 s is an approximation)

5



(a) $E_k = \frac{1}{2} m v^2 = \frac{1}{2} \times 2 \times 5^2$
$= \underline{\underline{25}}$ J

(b) $E_p = m g h = 2 \times 9.8 \times 3.2$
$= \underline{\underline{62.7}}$ J

(c) Change in E_k = change in gravitational E_p

E_k at bottom - E_k at top = mgh

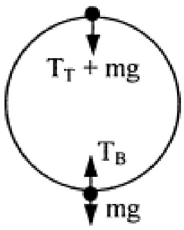
E_k at bottom - 25 = 62.7

E_k at bottom = $\underline{\underline{87.7}}$ J

(d) $87.7 \text{ J} = \frac{1}{2} m v^2$

$$v = \sqrt{\frac{2 \times 87.7}{2}} = \sqrt{87.7} = \underline{\underline{9.36}} \text{ m s}^{-1}$$

(e)



At the top, central force = $T_T + mg = \frac{m v^2}{r}$

hence $T_T = \frac{m v^2}{r} - mg = \frac{2 \times 5^2}{1.6} - 2 \times 9.8 = \underline{\underline{11.7}}$ N

At the bottom, central force = $T_B - mg = \frac{m v^2}{r}$

thus $T_B = \frac{2 \times 9.36^2}{1.6} + 2 \times 9.8 = 109.5 + 19.6 = \underline{\underline{129}}$ N

(f) For this to occur; the force of gravity should **just** equal $\frac{m v^2}{r}$

$mg = \frac{m v^2}{r}$ and $v = \sqrt{g\,r} = \sqrt{9.8 \times 1.6}$ $v = \underline{\underline{3.96}}$ m s^{-1}

(In this question all answers are given to three significant figures.)

6 Again $mg = \frac{m v^2}{r}$

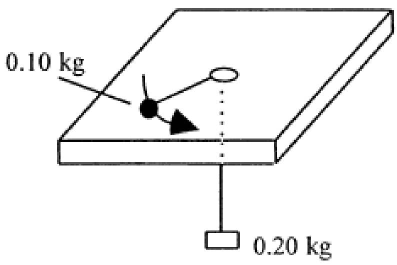
$v = \sqrt{g\,r} = \sqrt{9.8 \times 20}$ $v = \underline{\underline{14}}$ m s^{-1}

7 (a) Once more, the gravitational force supplies the central force:

$mg = \frac{m v^2}{r}$ and $v = \sqrt{g\,r} = \sqrt{9.8 \times 1.2} = \underline{\underline{3.4}}$ m s^{-1}

(b) $\omega = \frac{v}{r} = \frac{3.4}{1.2} = \underline{\underline{2.8}}$ rad s^{-1}

8



The tension in the string supplies the central force. The hanging mass supplies the tension.

$$\text{tension} = m\,g = 0.20 \times 9.8 = 1.96\ \text{N}$$

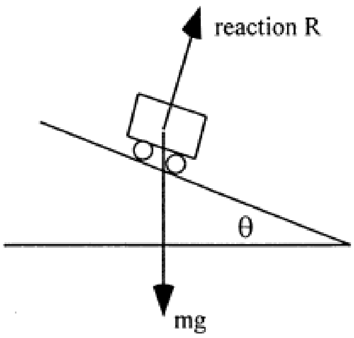
$$1.96 = \frac{m\,v^2}{r} = m\omega^2 r$$

$$\text{thus } \omega^2 = \frac{1.96}{0.10 \times 0.15}$$

$$\omega = 11.43\ \text{rad s}^{-1}$$

$$\text{thus r.p.m.} = \frac{11.43}{2\pi} \times 60 = \underline{109}\ \text{r.p.m.}$$

9 (a)



(b) In vertical direction there is no acceleration $R\cos\theta = mg$

In radial direction there is a central acceleration $R\sin\theta = \frac{m\,v^2}{r}$

Dividing the two equations gives $\tan\theta = \frac{v^2}{g\,r}$

$$\tan\theta = \frac{20^2}{9.8 \times 60}$$

$$\underline{\underline{\theta = 34^\circ}}$$

10 (a) Horizontally: central force $\frac{m\,v^2}{r} = T_f \sin 30^\circ$

Vertically: $mg = T_f \cos 30^\circ$

[*Notice that the upper angle is 30°*]

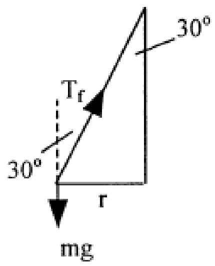
also $v = \frac{2\pi r}{T}$ where T is period of pendulum

note that $r = 1.2 \times \sin 30^\circ = 0.6$

dividing the above equations

$$\frac{v^2}{g\,r} = \frac{\sin 30}{\cos 30} \quad \text{or} \quad \frac{4\pi^2}{gT^2}\,r = \tan 30$$

$$T^2 = \frac{4\pi^2 r}{g \tan 30} = \frac{4\pi^2 \times 0.6}{9.8 \times \tan 30} \qquad \text{giving } T = \underline{2.05}\ \text{s}$$



(b) $v = \frac{2\pi r}{T} = \frac{2\pi \times 0.6}{2.05} = \underline{1.84}\ \text{m s}^{-1}$

TUTORIAL 4.0

Torque and moment of inertia

1. (a) The moment of a force is the turning effect of a force about an axis. The magnitude of the moment of a force is the force multiplied by the perpendicular distance from the direction of the force to the turning point.
   (b) Using a long handled screw driver to lever off a paint tin lid.
   Using a bottle opener to remove the cap from a bottle.

2. (a) T = F x r where F is the tangential force and r is the perpendicular distance from the direction of F to the axis of rotation.
   (b) T = I α where α is the angular acceleration and I is the moment of inertia.

3. The mass of the object and the distribution of the mass about a fixed axis.

4. The moment of inertia depends on the distribution of the mass about an axis. The contribution of each particle of mass in the rod to the moment of inertia is mr^2 where r is the perpendicular distance to the axis. For the rod about the axis through its centre more of the mass is at smaller distances from the axis so this arrangement will have the smaller moment of inertia.

   (For interest $I_{rod(centre)} = \frac{ML^2}{12}$ and $I_{rod(end)} = \frac{ML^2}{3}$.)

5. $I = mr^2 = MR^2 = 2 \times 0.8^2 = 1.28$ kg m^2
   Assuming all the 2.0 kg mass (M) is all at a distance of 0.80 m (R) from the axis.
   The mass of the spokes has been neglected, given as very light.

6. (a) T = F x r = 8 x 0.3 = 2.4 N m
   (b) T = I α thus 2.4 = 0.4 x α and α = 6 rad s^{-2}
   (c) $\omega^2 = \omega_0^2 + 2\alpha\theta$, $\omega^2 = 0 + 2 \times 6 \times \frac{5}{0.3}$ since $\theta = \frac{5}{\text{circumference}} \times 2\pi$
   ω = 14 rad s^{-1}.

7. (a) $I = mr^2 = 0.25 \times 0.2^2 = 0.01$ kg m^2 (all the mass is at 0.2 m from the axis.)
   (b) T = I α , T = 0.01 x 5 giving T = 0.05 N m

8. (a) $I_{Earth} = 0.4 \times 6.0 \times 10^{24} \times (6.4 \times 10^6)^2 = 9.8 \times 10^{37}$ kg m^2
   Assume the Earth is a sphere of uniform density.
   (b) Earth makes one revolution (2π radians) in 24 x 60 x 60 s
   hence □ = 7.272 x 10^{-5} rad s^{-1}
   v = r□ = 6.4 x 10^6 x 7.272 x 10^{-5} = 465 m s^{-1}
   (It is interesting to convert this to miles per hour to get a feel for the speed!)

9. (a) Moment of inertia of roundabout and child P = 500 + 50 x 1.25^2 = 578 kg m^2
   T = Iα 200 - 25 = 578 x α, α = 0.303 rad s^{-2}
   Maximum angular velocity $\omega = \omega_0 + \alpha t$, ω = 0 + 0.303 x 3 = 0.91 rad s^{-1}
   (b) When decelerating -T = Iα 25 = 578 x α, α = -0.0433 rad s^{-2}
   $\omega = \omega_0 + \alpha t$, 0 = 0.91 – 0.0433 x t, t = 21 s

TUTORIAL 5.0

Angular momentum and rotational kinetic energy

1. (a) In the absence of external torques, the resultant angular momentum of a rigid body before impact is equal to the resultant angular momentum after impact..
   (b) $L = I\omega$
   (c) $E_{rot} = ½ I\square^2$

2. $\omega = 120 \times \frac{2\pi}{60} = 4\pi$, $L = I\omega = 0.25 \times 4\pi$, $L = 0.020$ kg m^2 s^{-1}

3. (a) $\omega = 40 \times \frac{2\pi}{60}$, $v = r\omega$ $v = 20 \times 40 \times \frac{2\pi}{60} = 84$ m s^{-1}
   (b) $L = mvr = 2 \times 84 \times 20 = 3.4 \times 10^3$ kg m^2 s^{-1} (or use $L = I\omega$ with $I = 2 \times 20^2$)
   (c) zero

4. (a) Total angular momentum before = total angular momentum after
   $20 \times 10 \times \frac{2\pi}{60}$ = total angular momentum after
   angular momentum after shafts are locked together = 21 kg m^2 s^{-1}
   (b) $L = I\omega$, $21 = (20 + 30) \times \omega$, $\omega = 0.42$ rad s^{-1}

5. Torque, angular velocity, tangential force, angular acceleration.

6. (a) Moment of inertia of roundabout = $½ MR^2 = ½ \times 250 \times 1.5^2 = 281$ kg m^2
   Total moment of inertia = $281 + (40 \times 1.5^2) + (60 \times 0.75^2) = 405$ kg m^2
   (b) $E_k(rot) = ½ I\omega^2 = ½ \times 405 \times \left(35 \times \frac{2\pi}{60}\right)^2 = 2.7$ kJ

7. (a) $E_k(rot) = ½ I\omega^2 = ½ \times 2.5 \times 2^2 = 5$ J
   (b) New $E_k(rot) = ½ \times 2.5 \times 15^2 = 281$ J Energy required = 281 – 5 = 276 J

8. (a) The hollow cylinder has the larger moment of inertia. The moment of inertia depends on the square of the distance of the mass from the axis of rotation. For the hollow cylinder most of the mass is at a distance equal to the radius. For the solid cylinder much of the mass is at a distance less than the radius.
   (b) No. The length of the cylinder does not affect the distribution of the mass about this axis. Only the radius of the cylinder will affect the moment of inertia of each cylinder.

9. (a) $E_p = mgh = 3 \times 9.8 \times 0.3 \sin 40 = 5.7$ J (change in height = 0.3 sin40)
   (b) Change in E_p = change in E_k(linear) + change in E_k(rot)
   $5.7 = ½ m v^2 - 0 + ½ I \square^2 - 0$ (But $I = ½ MR^2$ and $v = R\omega$)
   $5.7 = ½ M v^2 + ½ \times (½ MR^2) \times \frac{v^2}{R^2}$
   $= M v^2 \{ ½ + (½ \times ½) \} = v^2 (3 \times 0.75)$
   $v = 1.6$ m s^{-1} (Notice how the equation can be simplified)

10. Total angular momentum before = total angular momentum after

$$I \times 40 \times \frac{2\pi}{60} + 0 = \{(0.05 \times 0.08^2) \times 33 \times \frac{2\pi}{60}\} + I \times 33 \times \frac{2\pi}{60}$$

$$I\,(40 - 33) = 0.05 \times 0.08^2 \times 33$$

$$I = 1.5 \times 10^{-3} \text{ kg m}^2$$

11. Total angular momentum before = total angular momentum after

$$5.0 \times 10^{-3} \times 3 + 0 = (5.0 \times 10^{-3} + 0.2 \times 0.06^2)\,\omega$$

$$\omega = 2.6 \text{ rad s}^{-1}$$

12. (a) Total angular momentum before = total angular momentum after

$$1.5 \times 30 = 10 \times \omega$$

$$\omega = 4.5 \text{ rad s}^{-1}$$

(b) Initially kinetic energy = $\frac{1}{2} I \omega^2$ = $\frac{1}{2} \times 1.5 \times 30^2$ = 675 J
Final kinetic energy $\frac{1}{2} \times 10 \times 4.5^2$ = 101 J
Change in kinetic energy = 574 J

(c) The skater will have supplied some energy to move her arms.

13. (a) The moment of inertia is altered when the distribution of the mass about the axis alters. Angular momentum is conserved in the absence of external forces. $L = I\,\omega$, hence if the moment of inertia is reduced the rotational speed must increase for L to remain unchanged.

(b) Total angular momentum before = total angular momentum after

$$5.0 \times 3.0 = I \times 25$$

$$I = 0.6 \text{ kg m}^2$$

14. Total angular momentum before = total angular momentum after

$$0 = 3 \times (20 \times 2^2) \times \frac{3}{2} + 300 \times \square$$

ω = − 1.2 rad s^{-1} (Notice that the roundabout moves 'backwards')

15. Total angular momentum before = total angular momentum after

$$I \times 100 \times \frac{2\pi}{60} = 0.02 \times 0.05^2 \times 75 \times \frac{2\pi}{60} + I \times 75 \times \frac{2\pi}{60}$$

$$I = 1.5 \times 10^{-4} \text{ kg m}^2$$

16. (a) $v = r\omega = 20 \times 10^3 \times 30 \times 2\pi = 3.8 \times 10^6 \text{ m s}^{-1}$

(b) $$\text{density} = \frac{\text{mass}}{\text{volume}} = \frac{1.99 \times 10^{30}}{\frac{4}{3}\pi(20 \times 10^3)^3} = 5.9 \times 10^{16} \text{ kg m}^{-3}$$

(c) Volume of 1 kg of star = $1/(5.9 \times 10^{16})$ = 1.7×10^{-17} m^3
Volume of star for 1.675×10^{-27} kg (a neutron) = 2.8×10^{-44} m^3
Assuming a cubic cell, length is cube root of volume, length = 3×10^{-15} m
Average spacing is approximately 1.5 m (diameter of neutron is 2×10^{-15} m)
(Notice that this is very small – consider the atomic size)

TUTORIAL 5.1

Torque, Moments of Inertia and Angular Momentum

1 (a) $T = I\alpha$ ($T = 0.8$ N m and $I = 1.2$ kg m^2)

$\alpha = \frac{T}{I} = \frac{0.8}{1.2} = 0.667$

$\alpha = \underline{0.67}$ rad s^{-2}

(b) $\omega = \omega_o + \alpha t$ $\omega_o = 0$ rad s^{-1} and $t = 5$ s

$= 0 + 0.667 \times 5$

$\omega = \underline{3.3}$ rad s^{-1}

2 (a) Change in E_p = change in $E_{k(rot)}$ flywheel + change in $E_{k(linear)}$ weight

$$mgh = \frac{1}{2} I \omega^2 + \frac{1}{2} m v^2$$

$$0.1 \times 9.8 \times 2 = \frac{1}{2} I \times \frac{v^2}{r^2} + \frac{1}{2} \times 0.1 \times v^2$$

$$2 \times 0.1 \times 9.8 \times 2 = I \times \frac{v^2}{0.1^2} + 0.1 \times v^2$$

$$3.92 = v^2 \times 100 I + 0.1 \times v^2 = v^2 (0.1 + 100 I)$$

$$v^2 = \frac{3.92}{0.1 + 100 I}$$

thus $v = \sqrt{\frac{3.92}{0.1 + 100 I}}$

(b) Now assume that the 0.10 kg mass falls with a uniform acceleration and that $u = 0$ m s^{-1}

In the vertical plane: $s = \frac{v + u}{2} t$

$$2 = \frac{v + 0}{2} \times 8$$

velocity of mass at the bottom $v = 0.5$ m s^{-1}

using the expression determined above $v^2 = \frac{3.92}{0.1 + 100 I}$

gives $(0.5)^2 = \frac{3.92}{0.1 + 100 I}$ and $100 I + 0.1 = \frac{3.92}{0.5^2}$

and $I = \frac{1}{100} (\frac{3.92}{0.5^2} - 0.1)$

$I = \underline{0.156}$ kg m^2

Alternatively the terms, mgh, and $\frac{1}{2}$ m v^2 can be calculated separately and the energy relationship used to determine I from the $\frac{1}{2} I \omega^2$ term.

3 (a) Frictional torque $T = r \times F = -5 \times 0.50 = -2.5$ N m

$$T = I\alpha$$

$$\alpha = \frac{-2.5}{2.0} = -1.25 \text{ rad s}^{-2}$$

using $\omega = \omega_o + \alpha t$ ($\omega_o = 10$ rev s^{-1} = 20π rad s^{-1})

$$0 = 20\pi - (1.25 \times t)$$

$$t = \frac{20\pi}{1.25} = \underline{\underline{50.3}} \text{ s}$$

(b) Use $\omega^2 = \omega_o^2 + 2\alpha\theta$

$$0 = (20\pi)^2 - 2 \times 1.25 \times \theta$$

$$\theta = \frac{(20\pi)^2}{2.5} = \underline{\underline{1.58 \times 10}}^3 \text{ rad} \quad (= 251 \text{ rev})$$

(c) Work = $T\theta = 2.5 \times 1580$

= $\underline{\underline{3.95}}$ kJ (= heat generated)

[*Note: calculations involving work done are* ***not*** *specified in the syllabus.*]

4 (a) Assume all the mass is at the rim, i.e. the spokes have negligible mass.

Moment of inertia = MR^2 all parts of the rim are at the same radius R

$$I = 2.0 \times (0.50)^2$$

$$I = \underline{\underline{0.50}} \text{ kg m}^2$$

(b) (i) Driving torque $T = r \times F$

$= 0.50 \times 20$

$= \underline{\underline{10}}$ N m

(ii) $T = I\alpha$

$$\alpha = \frac{T}{I} = \frac{10}{0.5} = \underline{\underline{20}} \text{ rad s}^{-2}$$

(c) (i) $\theta = \omega_o t + \frac{1}{2}\alpha t^2$

$= 0 + \frac{1}{2} \times 20 \times (5.0)^2$

$= \underline{\underline{250}}$ rad

(ii) Angular momentum, $L = I\omega$

$= 0.50 \times (\alpha t)$

$= 0.50 \times 100$

$L = \underline{\underline{50}}$ kg m^2 s^{-1}

(iii) $E_k = \frac{1}{2} I\omega^2$

$= \frac{1}{2} \times 0.50 \times 100^2$

$= \underline{\underline{2.5}}$ kJ

5 (a) $L = m v r = m \omega r^2$

$= 0.20 \times 2\pi \times 0.40^2$

Thus angular momentum of the mass = $\underline{\underline{0.20}}$ kg m^2 s^{-1}

(b) $v = \omega r = 2\pi \times 0.40 = 2.51$ m s^{-1}

$E_k = \frac{1}{2} m v^2 = \frac{1}{2} \times 0.20 \times 2.51^2$ (or use $E_k = \frac{1}{2} I \omega^2$)

$= \underline{\underline{0.63}}$ J

(c) Total angular momentum before = Total angular momentum after

$m_1 \omega_1 r_1^2 = m_1 \omega_2 r_2^2$ m_1 cancels

$2\pi \times 0.40^2 = \omega_2 \times 0.20^2$

$$\omega_2 = 2\pi \times \frac{(0.40)^2}{(0.20)^2}$$

$\omega_2 = \underline{\underline{8\pi}}$ rad s^{-1} (= 25 rad s^{-1})

(d) New $E_k = \frac{1}{2} m v^2 = \frac{1}{2} \times 0.2 \times 0.2^2 \times 25^2$ (or use $E_k = \frac{1}{2} I \omega^2$)

$= \underline{\underline{2.5}}$ J

The increase in energy was supplied when moving the object inwards.
The 'push' would have to be radial, so that no external torque is given.

6 (a) Total moment of inertia $= \frac{1}{12} M_{rod} L^2 + 2 \times [M_{mass} R^2]$

$= \frac{1}{12} \times 1.2 \times 1.0^2 + 2 \times [0.5 \times (0.5)^2]$

$= 0.10 + 0.25$

$I_{tot} = \underline{\underline{0.35}}$ kg m^2

(b) (i) $T = r \times F$

$= 0.50 \times 10$

$= \underline{\underline{5.0}}$ N m

(ii) $T = I\alpha$

$5.0 = 0.35\,\alpha$

$$\alpha = \frac{5.0}{0.35} = \underline{\underline{14.3}} \text{ rad s}^{-2}$$

(iii) $\omega = \omega_o + \alpha t = 0 + 14.3 \times 4.0 = 57.2$ rad s^{-1}

$E_k = \frac{1}{2} I \omega^2 = \frac{1}{2} \times 0.35 \times 57.2^2$

$= \underline{\underline{573}}$ J

7 Driving Torque = Frictional Torque (since constant angular velocity)

$T = I\alpha$ and $7.7 = 1.5 \times \alpha$

$$\alpha = \frac{7.69}{1.5} = 5.13 \text{ rad s}^{-2}$$

$\omega = \omega_o - \alpha t$ (deceleration caused by friction)

$0 = 52 - (5.13 \times t)$

$t = \underline{\underline{10.1}}$ s

8 (a) The solid cylinder will reach the bottom of the slope first.

(b) Change in gravitational E_p = change in rotational E_k + change in linear E_k

$$mgh = \frac{1}{2} I\omega^2 + \frac{1}{2} m v^2 = \frac{1}{2} I\left(\frac{v^2}{r^2}\right) + \frac{1}{2} m v^2$$

$$mgh = \frac{1}{2} v^2 \left(\frac{I}{r^2} + m\right)$$

$$v^2 = \frac{2mgh}{\left(\frac{I}{r^2} + m\right)} = \frac{2mgh}{\left(\frac{I + mr^2}{r^2}\right)} = \frac{2mghr^2}{I + mr^2}$$

The two cylinders have the same values for m, g,h and r. From the equation, if the moment of inertia increases then v^2 decreases and the speed v decreases. The solid cylinder has most of its mass at a radius less than the radius of the cylinder. The hollow cylinder has the *greater* moment of inertia because all the mass is at the radius of the cylinder, hence $v_{hollow} < v_{solid}$. Thus the solid cylinder reaches the bottom first because its linear speed will be greater.

9 (a) (i) solid cylinder: $I = \frac{1}{2} M R^2 = \frac{1}{2} \times 10 \times (0.10)^2$
$= \underline{\underline{0.05}}$ kg m^2

(ii) hollow cylinder: $I = \frac{1}{2} M (R^2 + r^2) = \frac{1}{2} \times 10 \times (0.10^2 + 0.05^2)$
$= \underline{\underline{0.0625}}$ kg m^2

(b) $M g h = \frac{1}{2} I \omega^2 + \frac{1}{2} M v^2$ but $\omega^2 = \frac{v^2}{R^2}$

$$M g h = \frac{1}{2} I \frac{v^2}{R^2} + \frac{1}{2} M v^2$$

$$v^2 = \frac{2 M g h R^2}{(I + M R^2)}$$ divide top and bottom by $M R^2$

$$v^2 = \frac{2gh}{\left[1 + \frac{I}{MR^2}\right]} \quad \text{and} \quad v = \sqrt{\frac{2gh}{\left[1 + \frac{I}{MR^2}\right]}}$$

(c) solid cylinder: $v = \sqrt{\frac{2 \times 9.8 \times 0.04}{\left[1 + \frac{0.05}{10 \times 0.10^2}\right]}} = \underline{\underline{0.723}}$ m s^{-1}

hollow cylinder: $v = \sqrt{\frac{2 \times 9.8 \times 0.04}{\left[1 + \frac{0.0625}{10 \times 0.10^2}\right]}} = \underline{\underline{0.695}}$ m s^{-1}

Time for a cylinder to roll down slope is given by $s = \frac{v + u}{2} t$ where u = 0

Time for solid cylinder to roll down the slope = 5.53 s.
Time for hollow cylinder to roll down the slope = 5.76 s.
Solid cylinder arrives at the bottom $\underline{\underline{0.23}}$ s ahead of the hollow cylinder.